Two and three-fermion 3D equations deduced from Bethe-Salpeter equations.

J. Bijtebier*
Theoretische Natuurkunde, Vrije Universiteit Brussel,
Pleinlaan 2, B1050 Brussel, Belgium.
Email: jbijtebi@vub.ac.be

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Abstract

We write a 3D equation for three fermions by combining the three two-body potentials obtained in 3D reductions (based on a series expansion around a relative-energy fixing "approximation" of the free propagators) of the corresponding two-fermion Bethe-Salpeter equations to equivalent 3D equations, putting the third fermion on its positive-energy mass shell. In this way, the cluster-separated limits are exact, and the Lorentz invariance / cluster separability requirement is automatically satisfied, provided no supplementary approximation, like the Born approximation, is made. The use of positive free-energy projectors in the chosen reductions of the two-fermion Bethe-Salpeter equations prevents continuum dissolution in our 3D three-fermion equation. The potentials are hermitian below the inelastic threshold and depend only slowly on the total three-fermion energy. The one high-mass limits are approximately exact.

This "hand-made" three-fermion 3D equation is also obtained by starting with an approximation of the three-fermion Bethe-Salpeter equation, in which the three-body kernel is neglected and the two-body kernels approached by positive-energy instantaneous expressions, with the spectator fermion on the mass shell. The neglected terms are then transformed into corrections to the 3D equation, in three steps implying each a series expansion. The result is of course complicated, but the lowest-order contributions of these correction terms to the energy spectrum remain manageable.

We also present some other 3D reduction procedures and compare them to our's: use of Sazdjian's covariant approximation of the free propagator, 3D reductions performed by a series expansion around instantaneous approximations of the kernels instead of "approximations" of the propagators, Gross' spectator model.

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*Senior Research Associate at the Fund for Scientific Research (Belgium).

Contents

1 Introduction

In the treatment of the three-body problem, there is a large gap between nonrelativistic quantum mechanics (Schrödinger equation) and relativistic quantum field theory (Bethe-Salpeter equation [1, 2]). Starting from the Schrödinger equation one can of course replace the free part of the hamiltonian by its relativistic form. Starting from the Bethe-Salpeter equation, one can try to eliminate the two relative time variables to get finally a Schrödinger equation with a compact potential plus a lot of correction terms of various origins. Instead of the Schrödinger equation, one can use the Faddeev equations. These equations can be derived from the Schrödinger equation, or, in a more general form, from the Bethe-Salpeter equation. Faddeev equations for the transition operator give the various scattering and reaction matrix elements and the poles (to be computed by basically nonperturbative methods) of this operator in the total energy give the spectrum of the three-body bound states.

Schrödinger's equation is "relatively easy" to solve, but this zero-order approximation does not reflect several important properties and symmetries of the studied physical system. These could be recovered, in principle, by incorporating higher-order contributions (in general an infinity of them).

As an intermediate step between nonrelativistic quantum mechanics and quantum field theory we shall search for a 3D equation built with the sum of the relativistic free hamiltonians plus two-body and perhaps also three-body interaction potentials. Such equations are closely related to the systems of coupled Dirac or Klein-Gordon equations of constraint theory [3-7], which are still 4D equations but exhibit a simplified dependence in the relative times, which could be completely eliminated to get a single

3D equation. Our 3D equation will be a strategic link in a chain of approximations between a manageable equation (such as Schrödinger's equation) and a full 3D equivalent (at least for what concerns the bound states spectrum) of the original Bethe-Salpeter equation. We shall try to keep a maximum of properties and symmetries of the original Bethe-Salpeter equation in our 3D equation. For the properties which we shall be unable to satisfy exactly in our 3D equation, we expect that the inclusion of higher-order correction terms will progressively improve the situation. We shall thus try to make our 3D equation satisfy the following list of requirements:

— Correct nonrelativistic limit.

— Lorentz invariance and cluster separability. It is always possible to render an equation Lorentz invariant by working in the general rest frame (center of mass reference frame) and by building invariants with the total 4-momentum vector, although the result may be unelegant and artificial. The Lorentz invariance requirement becomes a tool when combined with the cluster separability requirement: when all mutual interactions are "switched off", we must get a set of three free Dirac equations. This total separability can easily be obtained by using as hamiltonian the sum of three free Dirac hamiltonians and interaction terms. The real difficulty appears when only the interactions with fermion 3 (for example) are switched off. If we want a full cluster separability, the resulting equation for the (12) cluster can not refer to the global center of mass frame anymore, as the momentum of fermion 3 enters in the definition of this frame.

Lorentz invariance and cluster separability can be explicit or implicit (via rearrangements). The best known example of implicit Lorentz invariance is a free Dirac equation solved with respect to the energy: it becomes explicitly covariant by multiplication with the β matrix. Other implicit Lorentz invariances are not that trivial. For example, the 3D reductions of a Bethe-Salpeter equation are implicitly covariant, provided the series generated by this reduction is not truncated.

In fact, the homogeneous Bethe-Salpeter equation from which we shall start does not obey cluster separability: this equation is valid for three-fermion bound states only, i.e. for a total energy below all the continua. Our 3D equation will thus be used in the computation of three-fermion bound states. A priori, it will not necessarily give also the correct scattering amplitudes or obey cluster separability, even with all higher-order terms included. We shall nevertheless try to get a 3D equation which is also valid in the continuum region.

— Hermiticity and total energy independence of the interaction terms. In the two-body problem, nonhermitian interaction terms can be hermitian with respect to a modified scalar product, or made hermitian via a rearrangement of the equation. In the three-body problem these rearrangements could be more complicated. The hermiticity and the independence on the total energy are linked features, as one of these is often achieved at the expense of the other one. Energy depending interaction terms destroy some of the advantages of the use of an hermitian hamiltonian, such as the mutual orthogonality of the solutions, and leads to modify the usual perturbation calculation methods. The 3D potentials deduced from field theory are generally energy dependent, at least in the higher-order terms. We shall require hermiticity and energy-independence (or slow energy dependence) below the inelastic threshold in the lowest order terms at least.

— Correct heavy mass limits. When the mass of one of the fermions becomes infinite, its presence must be translated in the equations by a potential (Coulombian in QED) acting on the other fermions. For the two-body problem the "one-body limit" in QED is indeed a Dirac-Coulomb equation or a rearrangement of it (for example when a projection operator is introduced in order to avoid continuum dissolution - see below). In the Dirac-Coulomb equation, the Coulomb potential is already given by the limit of the Born term, as the higher-order crossed and ladder terms cancel mutually at the one-body limit. In contrast, the one-body limits of the rearrangements contain contributions from all terms [8-12].

— Perturbative approach. It must be possible to start with a manageable approximate equation and indefinitely improve the approximation of the measurable quantities (with respect to the uncalculated predictions of the here assumed exact Bethe-Salpeter equation) by adding higher-order contributions.

— Solution of the continuum dissolution problem. In the relativistic equations for several relativistic particles, the physical bound states are degenerate with a continuum of states combining asymptotically free particles with opposite energy signs. This often neglected fact forbids the building of normalizable solutions in the $N > 2$-body problem (including the two-body plus potential problem. In the pure two-body problem the mixing is prevented by the conservation of the total momentum). The usual solution consists in including positive-energy projection operators into the zero-order propagator [13-16]. The modified equations must of course continue to satisfy the other requirements, like the Lorentz invariance

/ cluster separability requirement.

Among these requirements, the absence of continuum dissolution is a technical necessity, if we want to get normalizable bound states. The correct nonrelativistic limit is of course a must. The possibility of improving the chosen approximation could enable us to estimate the precision of this approximation and possibly to satisfy the other requirements up to the considered order at least.

Section 2 is devoted to the two-fermion problem, revisited in order to define the notations and present the building blocks and cluster-separated limits of our future three-fermion equations. In section 3 we write directly a cluster separable 3D equation, by combining the relativistic free hamiltonians and the 3D potentials obtained in the reduction of the two-fermion Bethe-Salpeter equations, putting in each 3D potential the third fermion on the mass shell. Switching off two of the three mutual interaction potentials gives then a free Dirac equation for a spectator fermion plus the two-body equation which would be obtained directly by 3D-reducing the two-fermion Bethe-Salpeter equation, for the two interacting fermions. This insures the Lorentz invariance / cluster separability property. Positive-energy projectors solve the continuum dissolution problem. Our 3D equation can also be obtained from the three-fermion Bethe-Salpeter equation, by neglecting the three-body kernel and replacing the two-body kernels by positive-energy instantaneous (i.e. independent of the relative energies) approximations, equivalent at the cluster-separated limits. The remaining dependence (in the free propagators) of the approximated Bethe-Salpeter equation in the relative energies can then be eliminated by performing closed path integrals. This derivation of our 3D equation from an approximated Bethe-Salpeter equation enabled us to transform the neglected terms, known at the Bethe-Salpeter equation level, into correction terms to the 3D equation. We achieved this by performing three consecutive expansions of the correction terms: the three-fermion Bethe-Salpeter equation is first transformed into a set of three coupled equations for three wave functions depending each on one two-fermion total energy, then into a set of three 3D equations and finally into a single 3D equation. The resulting expression for the 3D equation is rather complicated, as it combines three series expansions, but we use it to write a manageable expression of the first-order corrections to the energy spectrum. The contributions of the residues of the poles of the propagators to this energy shift are easy to compute in terms of the Bethe-Salpeter kernels with two of the three fermions on their mass shell, in the spirit of Gross' spectator model [17, 18]. Besides, we have also the contributions of the singularities of the kernels themselves to the closed path integrals. These contributions can not be a priori neglected. In section 4, we examine some other possible choices of the "approximation" of the free propagator, including Sazdjian's covariant propagator [6, 12, 19]. In combination with a covariant substitute of the projector on the positive free-energy states, this propagator leads to a 3D potential given by a series in which each term is separately covariant, so that we can truncate the series without breaking the covariance. In section 5, we examine an alternative 3D reduction method built around instantaneous approximations of the Bethe-Salpeter kernels. The non-hermitian resulting 3D potential can be made hermitian either by performing a second series expansion, easily combinable with the series expansion generated by the 3D reduction, or by adaptating the starting approximation of the Bethe-Salpeter kernel (Phillips and Wallace's method [20]). It is interesting to note that Phillips and Wallace's method and the propagator-based 3D reduction method give both the correct scattering operator (a bonus not implied by the starting homogeneous Bethe-Salpeter equation, designed for bound states), while the ordinary kernel-based 3D reduction does not. Section 6 is devoted to a comparison of our "hand-made" 3D equation with Gross' spectator model's equations [17, 18] in Faddeev's formalism. In this model, the dependence of the two-fermion transition operators in the relative energies is taken into account by putting different fermions on their positive-energy mass shells, according to the neighbours of each two-fermion transition operator in the expansion giving the three-fermion transition operator. Each of our two-fermion transition operators is everywhere approximated in the same way, so that our Faddeev equations are much simpler and can be transformated into our wave equation. The difference between both models corresponds to higher-order contributions, which we recover in our calculation of the higher-order terms, together with the supplementary contributions, not to be a priori neglected, of the singularities of the kernels to the closed path integrals in the relative energies complex planes. In section 7, we compare the one-high mass limits ("two-body limits") of our three-fermion equation with our previous two-fermion in an external potential equation [16]. Section 8 is devoted to conclusions.

2 The two-fermion problem.

2.1 Notations.

We shall write the Bethe-Salpeter equation for the bound states of two fermions [1] as

$$\Phi = G^0 K \Phi, \tag{1}$$

where Φ is the Bethe-Salpeter amplitude, function of the positions x_1, x_2 or of the momenta p_1, p_2 of the fermions, according to the representation chosen. The operator K is the Bethe-Salpeter kernel, given as a factor of the kernel of an integral equation in momentum space by the sum of the irreducible two-fermion Feynman graphs. The operator G^0 is the free propagator, given by the product $G_1^0 G_2^0$ of the two individual fermion propagators

$$G^0 = G_1^0 G_2^0, \qquad G_i^0 = \frac{1}{p_{i0} - h_i + i\epsilon h_i}\,\beta_i = \frac{p_{i0} + h_i}{p_i^2 - m_i^2 + i\epsilon}\,\beta_i \tag{2}$$

where the h_i are the Dirac free hamiltonians

$$h_i = \vec{\alpha}_i \,.\vec{p}_i + \beta_i\, m_i \qquad (i = 1, 2). \tag{3}$$

The Bethe-Salpeter kernel K should contain charge renormalization and vacuum polarization graphs, while the propagators G_i^0 should contain self-energy terms (which can be transferred to K [19, 21]]). In this work, we consider only the free fermion propagators in G_i^0 and the "skeleton" graphs in K, and we hope that the inclusion of the various corrections would not change our conclusions.
We shall define the total (or external, CM, global) and relative (or internal) variables:

$$X = \frac{1}{2}(x_1 + x_2)\ , \qquad P = p_1 + p_2\ , \tag{4}$$

$$x = x_1 - x_2\ , \qquad p = \frac{1}{2}(p_1 - p_2). \tag{5}$$

and give a name to the corresponding combinations of the free hamiltonians:

$$S = h_1 + h_2\ , \quad s = \frac{1}{2}(h_1 - h_2). \tag{6}$$

We know that, at the no-interaction limit, the system is described by a pair of free Dirac equations:

$$(p_{10} - h_1)\Psi = 0, \qquad (p_{20} - h_2)\Psi = 0, \tag{7}$$

where Ψ depends on x_1, x_2. Let us also write their iterated version

$$(p_{10}^2 - E_1^2)\Psi = 0, \qquad (p_{20}^2 - E_2^2)\Psi = 0 \tag{8}$$

with

$$E_i = \sqrt{h_i^2} = (\vec{p}_i^{\,2} + m_i^2)^{\frac{1}{2}}. \tag{9}$$

Interesting combinations can be obtained from the sum and differences of the equations (7) or of the iterated equations (8):

$$(P_0 - S)\Psi = 0, \qquad (p_0 - s)\Psi = 0, \tag{10}$$

$$H^0\Psi = 0, \qquad (p_0 - \mu)\Psi = 0 \tag{11}$$

with

$$H^0 = 2[(p_1^2 - m_1^2) + (p_2^2 - m_2^2)]_{p_0=\mu} \ = \ P_0^2 - 2(E_1^2 + E_2^2) + 4\mu^2, \tag{12}$$

$$\mu = \frac{1}{2P_0}(E_1^2 - E_2^2) = \frac{1}{2P_0}(h_1^2 - h_2^2) = \frac{sS}{P_0}. \tag{13}$$

The choice (4,5) of the variables X and p was partially arbitrary: other linear combinations are possible, provided the canonical commutation relations are satisfied. In the nonrelativistic framework, the choice of $\vec{X}$ is weighted by the masses, and $\vec{p}$ is chosen accordingly. A frequent choice in the relativistic framework is a weighting by the free energies, which leads to a simpler complementary equation $p_0\Psi = 0$ instead

of (11), but has the drawback that the coefficients of the weighting are not constant. We think that our choice (4,5) leads to simpler expressions, even if taking the nonrelativistic limit is less straightforward.

Our free equations (10) are eigenvalue equations, the energies P_0 and p_0 being the eigenvalues and S and s the corresponding operators. Equation (11) with H^0 is a fourth-order equation in P_0, with four solution for a given set of values of the spatial momenta. Equation (11) with p_0 is the eigenvalue equation of an operator μ, depending itself on P_0. In the Bethe-Salpeter equation (1), P_0 remains a constant, but p_0 becomes an operator in position space, or a variable in momentum space. The 3D reduction will consist in eliminating this variable by performing an expansion around a fixed combination of the other ones (s or μ in the following).

2.2 3D reduction of the two-fermion Bethe-Salpeter equation.

In the zero-order approximation, the free propagator G^0 will be replaced by a carefully chosen expression G^δ, combining a constraint like $\delta(p_0-\mu)$ fixing the relative energy, and a global 3D propagator like $-2i\pi(P_0-S)^{-1}\beta_1\beta_2$. The argument of the δ and the inverse of the propagator should be combinations of the operators used in the free equations (at last approximately and for the positive-energy solutions). There exists an infinity of possible combinations [8, 12, 20, 22-34]. The best choice depends on the quantities one wants to compute (energy of the lowest state, hyperfine splitting, recoil of a nucleus, etc...) and on the properties one wants to preserve exactly in the first approximation (cluster separability, Lorentz invariance, heavy mass limits, charge conjugation symmetry...). All choices would be equivalent if all correction terms could be computed, but this is of course impossible.

We shall see below that the reduced wave function Ψ (from which the relative time-energy degree of freedom can be trivially eliminated) is given by

$$\Psi = G^\delta (G^0)^{-1}\Phi. \tag{14}$$

The choice of the constraint fixes the 3D hypersurface ($p_0=\mu$ for example) on which we want to work. The remaining of G^δ is a purely 3D operator, the different choices of which result in different 3D operators applied on a common basic 3D wave function and in different rearrangements of a common reduction series giving the 3D potential. The various propagator-based 3D reductions of the literature can thus be classified according to the constraint they use. Once this constraint chosen, we can only write different equivalent forms (or sometimes projections) of the same 3D equation. It is the unavoidable truncation of the reduction series which makes the difference (numeric, if we simply want to compute predictions of field theory; more fundamental, if we want to write constraint theory equations with QCD inspired potentials).

Two natural choices for the constraint are $\delta(p_0-s)$, based on the first-order equations (10) and $\delta(p_0-\mu)$, based on the second-order equations (11). In $\delta(p_0-s)$, s is an operator which can be diagonalized in momentum space using the four projectors on the subspaces corresponding to the different signs of h_1, h_2 (see below). By contrast, μ is a momentum-depending number, which reduces to $(m_1^2-m_2^2)/2P_0$ in the two-fermion center of mass frame $\vec{P}=0$ which can be defined in the pure two-fermion problem (but not in the two-fermion plus potential problem or in the three-fermion problem). Other constraints could also be chosen, such as that of Gross [8], which puts one particle (normally the heaviest) on its mass shell. We made a nonexhaustive review in ref. [12].

If we consider the contributions of the poles of

$$G^0 = \frac{1}{\frac{1}{2}P_0 + p_0 - h_1 + i\epsilon h_1}\,\frac{1}{\frac{1}{2}P_0 - p_0 - h_2 + i\epsilon h_2}\beta_1\beta_2 \tag{15}$$

in an expression like KG^0K, we must perform an integration with respect to p_0. If K is instantaneous, we get

$$\int dp_0 G^0(p_0) \;=\; -2i\pi\tau\, g^0\, \beta_1\beta_2, \qquad g^0 \;=\; \frac{1}{P_0 - S + i\epsilon P_0} \tag{16}$$

where

$$\tau = \frac{1}{2}(\tau_1+\tau_2), \qquad \tau_i = \frac{h_i}{\sqrt{h_i^2}} = \frac{h_i}{E_i} = \mathrm{sign}(h_i). \tag{17}$$

can also be written

$$\tau = \Lambda^{++} - \Lambda^{--}, \qquad \Lambda^{ij} = \Lambda_1^i\Lambda_2^j, \qquad \Lambda_i^\pm = \frac{E_i \pm h_i}{2E_i}. \tag{18}$$

When K is not instantaneous, we must add the contributions of its singularities. Furthermore, in the residues of the poles of G^0 we must take K at $p_{10}=h_1$ or at $p_{20}=h_2$, according to the chosen integration path and to the sign of τ. In order to preserve an explicit symmetry between the two fermions, we shall use the constraint $\delta(p_0-s)$, to be combined with g^0 :

$$G^{\delta S}(p_0) = -2i\pi\,\tau\,\delta(p_0-s)\,g^0\,\beta_1\beta_2. \tag{19}$$

The operator τ has a clear meaning in the basis built with the free solutions: it is +1 for $h_1, h_2 > 0$, -1 for $h_1, h_2 < 0$ and zero when they have opposite signs. It comes from the dependence of the p_0 integral on the signs of the $i\epsilon h_i$.

The choice $G^\delta = G^{\delta S}$ has two merits: a) It leads directly to Salpeter's equation without higher-order correction terms to the potential when the Bethe-Salpeter kernel is instantaneous. b) In the two-fermion plus potential problem and in the three-fermion problem, the operator τ prevents the mixing of asymptotically free fermions with opposite energy signs, which is the origin of the continuum dissolution problem (see section 4 below). In order to reduce the number of terms at lowest order, we shall however keep only the Λ^{++} part of τ and choose

$$G^\delta(p_0) = G^{\delta P}(p_0) = -2i\pi\,\Lambda^{++}\,\delta(p_0-s)\,g^0\,\beta_1\beta_2. \tag{20}$$

We shall examine some other possible choices in section 4.

Let us now write the free propagator as the sum of the zero-order propagator, plus a remainder:

$$G^0 = G^\delta + G^R. \tag{21}$$

The Bethe-Salpeter equation becomes then the inhomogeneous equation

$$\Phi = G^0 K\Phi = (G^\delta + G^R)K\Phi = \Psi + G^R K\Phi, \tag{22}$$

with

$$\Psi = G^\delta K\Phi \qquad (= G^\delta (G^0)^{-1}\Phi). \tag{23}$$

Solving (formally) the inhomogeneous equation (22) with respect to Φ and putting the result into (23), we get

$$\Psi = G^\delta K(1-G^R K)^{-1}\Psi = G^\delta K^T\Psi \tag{24}$$

where

$$K^T = K(1-G^R K)^{-1} = K + KG^R K + ... = (1-KG^R)^{-1}K \tag{25}$$

obeys

$$K^T = K + KG^R K^T = K + K^T G^R K. \tag{26}$$

The reduction series (25) re-introduces in fact the reducible graphs into the Bethe-Salpeter kernel, but with G^0 replaced by G^R. Equation (24) is a 3D equivalent of the Bethe-Salpeter equation.

The relative energy dependence of eq. (24) can be easily eliminated:

$$\Psi = \delta(p_0-s)\,\psi \tag{27}$$

and ψ obeys:

$$\psi\; = -2i\pi\,g^0\,\Lambda^{++}\int dp_0' dp_0 \delta(p_0'-s)\beta_1\beta_2 K^T(p_0', p_0)\delta(p_0-s)\,\psi. \tag{28}$$

Using the identity $\psi = \Lambda^{++}\psi$, we can write

$$\psi\; = g^0\,V\,\psi, \qquad V\; = -2i\pi\,\beta_1\beta_2 K^{T++}(s,s), \tag{29}$$

$$\beta_1\beta_2 K^{T++}(s,s)\;\equiv\;\Lambda^{++}\int dp_0' dp_0 \delta(p_0'-s)\beta_1\beta_2 K^T(p_0', p_0)\delta(p_0-s)\,\Lambda^{++}. \tag{30}$$

Note that we write (p_0', p_0) but (s,s), as we keep s in operator form. This operator can be diagonalized in the spatial momentum space by using the Λ^{ij} projectors. The eigenvalue will depend on the position of s in the formula: the eigenvalue of the first s in (30) will be built with the final momenta and that of the last s will be built with the initial momenta.

The inversion of the reduction is given by

$$\Phi \,=\, (\,1 - G^R K\,)^{-1}\,\Psi \,=\, (\,1 + G^R K^T\,)\,\Psi \,=\, (\,1 + G^0 K^T - G^{\,\delta} K^T\,)\,\Psi \,=\, G^0 K^T\,\Psi \tag{31}$$

or, explicitating the relative energy:

$$\Phi(p_0') \,=\, G^0(p_0')\,K^T(p_0', s)\,\psi. \tag{32}$$

The splitting of G^0 into two terms containing a δ is the origin of unphysical singularities in the terms of K^T when the argument of the delta vanishes on the singularities of K. When the full K^T is computed, the singularities of the different terms cancel mutually. When K^T is truncated, some of the unphysical singularities have to be removed by hand [8, 17].

This 3D reduction can also be described in terms of transition operators. The 4D transition operator is

$$T \,=\, K \,+\, K\,G^0\,K \,+\, \cdots \tag{33}$$

and K^T can be obtained by keeping only the G^R part of G^0 in it. We have also

$$\begin{aligned} T = K(1-G^0K)^{-1} &= K(1-G^RK-G^\delta K)^{-1} = K(1-G^RK)^{-1}(1-G^\delta K(1-G^RK)^{-1})^{-1} \\ &= K^T(1-G^\delta K^T)^{-1} = K^T + K^TG^\delta K^T + \cdots \end{aligned} \tag{34}$$

so that the 3D transition operator

$$T^{3D} \,=\, V \,+\, V\,g^0\,V \,+\, \cdots \tag{35}$$

is also given by

$$T^{3D} \,=\, -2i\pi\,\beta_1\beta_2\,T^{++}(s,s). \tag{36}$$

We see that the 3D transition operator is a constrained form of that of field theory. Both operators become equal to the physical scattering amplitude when both fermions are on their positive-energy mass shells. This was not guaranteed a priori, as our original two-fermion Bethe-Salpeter equation (1) was valid only for bound states. Our 3D equation (29) is a bound state equation too. To include the scattering states we should add an inhomogeneous term, or write the equation in the form

$$(P_0 - E_1 - E_2\,)\,\psi \,=\, V\,\psi. \tag{37}$$

Conversely:

$$\begin{aligned} T &= K^T + K^TG^\delta K^T + K^TG^\delta(K^T + K^TG^\delta K^T + \cdots)G^\delta K^T \\ &= K^T + K^TG^\delta K^T + K^TG^\delta\,\frac{-1}{2i\pi}\,\beta_1\beta_2\,T^{3D}G^\delta K^T. \end{aligned} \tag{38}$$

A Bethe-Salpeter equation leading directly to the same 3D reduction can be obtained by replacing the kernel K by the instantaneous positive-energy kernel $K^{T++}(s,s)$. In this case the 4D transition operator T becomes equal to $T^{++}(s,s) = -\beta_1\beta_2\,T^{3D}/2i\pi$.

In contrast with the textbook cases, the potentials deduced from field theory are in general energy-dependent (although one starts often with an energy-independent approximation). We shall thus get a set of eigenvalues $W^\lambda(P_0)$ corresponding to mutually orthogonal eigenstates $\psi^\lambda(P_0)$ of the hamiltonian. The physical energy spectrum will be given by the solutions of the algebric equations

$$P_0 \,=\, W^\lambda\,(P_0) \tag{39}$$

and the corresponding wave functions are no more orthogonal with the the usual scalar product, as they are no more eigenstates of the same hamiltonian. The usual methods of perturbation calculation are to be revisited [35].

The first-order correction term to the energy is

$$P_0 - P_0^{(0)} \,=\, -2i\pi \,<\Lambda^{++}\,\beta_1\beta_2\,(KG^RK)(s,s)\,\Lambda^{++}>\,. \tag{40}$$

Since $P_0^{(0)}$, $(KG^RK)(s,s)$ and the zero-order wave functions are in general total energy dependent, (40) is in fact a numerical equation in P_0 (or a matricial equation in case of degeneracy). If the P_0 dependence of the zero-order potential is of higher-order, one has only to take the P_0 dependence of $P_0^{(0)}$ into account.

What is lost in our 3D equation compared with the starting Bethe-Salpeter equation? The missing information is given by equation (32), which gives the bound state solutions of the Bethe-Salpeter equation in terms of the corresponding solutions of the 3D equation by adding the dependence in the relative energy. Furthermore, the transition operators are the same on the positive-energy mass shells, although the homogeneous Bethe-Salpeter equation was originally deduced from the inhomogeneous Bethe-Salpeter equation (for the propagators) by postulating a bound state [1]. Were some bound states lost in the 3D reduction, like the "abnormal solutions" related to excitations in the relative time degree of freedom? We have studied this still open problem elsewhere. Our conclusions were: 1). The abnormal solutions of the Bethe-Salpeter equation, when they exist, can also be obtained in the 3D equation via the total energy dependence of the potential, leading to multiple solutions of (39) for each value of λ [36, 37]. 2). The abnormal solutions found until now are artefacts due to the ladder approximation, and should not appear when all crossed graphs are included [37, 38].

3 The three-fermion problem.

3.1 A cluster-separable three-fermion 3D equation.

In the three-fermion framework, we shall append indices (ij)=(12),(23),(31) to the two-fermion symbols and write $P_{ij}, K_{ij},$ etc.... The index k=1,2,3 will denote the third fermion. Some of these symbols will also get a new three-fermion meaning when left unindexed, such as

$$P = p_1 + p_2 + p_3, \qquad S = h_1 + h_2 + h_3. \tag{41}$$

A cluster-separable three-fermion 3D equation can be obtained by combining three two-fermion potentials (29):

$$\psi = \frac{1}{P_0 - S} \, (V_{12} + V_{23} + V_{31}) \psi \tag{42}$$

given by our 3D reduction in the two-fermion framework. The potential V given by (29) depended on the total two-fermion energy P_0 which we did not wrote as an argument, because it was conserved. When using these two-fermion potentials in the three-fermion framework, we shall replace the now nonconserved two-fermion energy argument P_{ij0} by the operator $P_{ij0} + (p_{k0} - h_k) = P_0 - h_k,$ which is the difference between the conserved three-body energy and the free energy of the third fermion. The potential V_{ij} is thus no more strictly a two-fermion term, and the third fermion is no more strictly a spectator when there are interactions between the cluster (ij) and the fermion k. At the (ij)+k cluster-separated limit, however, the fermion k will be free and $P_0 - h_k$ will be replaced in V_{ij} by the now conserved P_{ij0}. Let us indeed "switch off" the potentials V_{23}^0 and V_{31}^0 in equation (42). This equation can then be splitted into

$$\psi = \psi_{12} \psi_3, \qquad P_0 = P_{120} + p_{30}, \tag{43}$$

$$\psi_{12} = \frac{1}{P_{120} - E_1 - E_2} V_{12}(P_{120}) \, \psi_{12}, \qquad (p_{30} - h_3) \, \psi_3 = 0, \tag{44}$$

i.e. equation (29) for the (12) subsystem and a free Dirac equation for fermion 3. Our 3D equation (42) satisfies thus clearly the cluster separability requirement. Furthermore, the three cluster-separated limits are exact equivalents of the corresponding two-fermion Bethe-Salpeter equations. This cluster separability is a property of the equation, or, more exactly, of the full Green function. For a given scattering solution it is also possible to take the cluster-separated limit at fixed P. This is not possible for the bound state solutions.

In replacing P_{ij0} by $P_0 - h_k$ we exploit the fact that the opposite cluster is at the cluster-separated limit a free fermion. With four fermions, we should introduce, for exemple, a potential V_{34} inside a potential V_{12} and vice-versa, getting an unmanageable infinite nesting of potentials. This limits our recipe to the case of three fermions.

Our two and three-fermion equations were written until now in an unspecified reference frame. We shall consider three possible answers to the covariance challenge:

– Write explicitly covariant equations. This is not the case ou our two and three-fermion 3D equations, built around non-covariant "approached" free propagators. We shall briefly present in section 4 a 3D reduction method based on Sazdjian's covariant "approached" free propagator [6, 12, 19].

– Choose the total rest frame $\vec{P}=\vec{0}$ as the reference frame. All quantities can then be made invariant by building invariants with the 4-vector P.

– Search for an implicit covariance by equivalence with an explicitly covariant equation. A 3D equation obtained by reduction of an explicitly covariant Bethe-Salpeter equation will for example be implicitly covariant if all terms of the series generated by the reduction are kept, even if the terms of the series are not separately covariant. If the series is truncated, the 3D equation will be (implicitly) covariant up to a given order only.

What about our equation (42)? We can write it in the three-fermion reference frame. If we "switch off" the potentials V_{23} and V_{31}, the equation for the subsystem (12) will still be written in the global three-fermion rest frame, but this link with fermion 3 will disappear if we go back to the original two-fermion Bethe-Salpeter equation, which is explicitly covariant. If the series generated by the reduction of this equation is truncated, we loose the exact covariance of the two-fermion equation and the exact cluster separability of the three-fermion equation.

A well known difficulty in the writing of equations for two mutually interacting relativistic particles in an external potential and for three or more mutually interacting relativistic particles is the continuum dissolution problem: if we combine asymptotically free particles with opposite energy signs, it becomes possible to build a continuum of solutions at any given total energy. This renders the building of normalisable bound state solutions impossible. The introduction of our positive free-energy projectors is the usual way of solving this problem. Our 3D equation has solutions built entierely in the $\Lambda^{+++}=+1$ subspace and satisfying

$$\psi = \frac{\Lambda^{+++}}{P_0 - E_1 - E_2 - E_3}\,(V_{12}+V_{23}+V_{31})\psi. \tag{45}$$

We shall come back to this point in section 4.

3.2 Positive-energies instantaneous appproximation of the three-fermion Bethe-Salpeter equation.

The Bethe-Salpeter equation for three fermions is

$$\Phi = \left[G_1^0 G_2^0 K_{12} + G_2^0 G_3^0 K_{23} + G_3^0 G_1^0 K_{31} + G_1^0 G_2^0 G_3^0 K_{123}\right]\Phi \tag{46}$$

where K_{123} is the sum of the purely three-body irreducible contributions. Let us first consider the case of three two-body positive-energy, instantaneous and cluster-energy independent interactions K_{ij}^0 (i.e. containing Λ_{ij}^{++} projectors and independent of $p'_{ij0}, p_{ij0}, P_{ij0}$). The Bethe-Salpeter equation becomes in this case

$$\Phi = \frac{-1}{2i\pi}\, G_1^0 G_2^0 G_3^0\, \beta_1\beta_2\beta_3\, \left[\, V_{12}^0\,\psi_{12} \,+\, V_{23}^0\,\psi_{23} \,+\, V_{31}^0\,\psi_{31}\,\right] \tag{47}$$

$$V_{ij}^0 \,=\, -2i\pi\,\beta_i\beta_j K_{ij}^0, \qquad \psi_{ij}(p_{k0}) = \beta_k\, G_{0k}^{-1} \int dp_{ij0}\,\Phi. \tag{48}$$

Replacing Φ in (48) by its expression (47) leads to a set of three coupled integral equations in the ψ_{ij} :

$$\psi_{12}(p_{30}) = \frac{-1}{2i\pi}\int dp_{120}\, G_1^0 G_2^0\, \beta_1\beta_2\, \left[\, V_{12}^0\,\psi_{12}(p_{30}) \,+\, V_{23}^0\,\psi_{23}(p_{10}) \,+\, V_{31}^0\,\psi_{31}(p_{20})\,\right] \tag{49}$$

where p_{10}, p_{20} must be written in terms of P_0, p_{30}, p_{120} :

$$p_{10} \,=\, \frac{P_0 - p_{30}}{2} \,+\, p_{120}, \qquad p_{20} = \frac{P_0 - p_{30}}{2} \,-\, p_{120}. \tag{50}$$

and similarly for ψ_{23} and ψ_{31}. We shall now search for solutions $\psi_{ij}(p_{k0})$ analytical in the lower $\mathrm{Im}(p_{k0}) < 0$ half plane and perform the integration (49) by closing the integration paths around these regions. The only singularities will then be the poles of the free propagators. The result is

$$\psi_{12}(p_{30}) = \frac{\Lambda_{12}^{++}}{(P_0 - S) - (p_{30} - h_3) + i\epsilon}\, \left[\, V_{12}^0\psi_{12}(p_{30}) + V_{23}^0\psi_{23}(E_1) + V_{31}^0\psi_{31}(E_2)\,\right] \tag{51}$$

and similarly for ψ_{23} and ψ_{31} (from now on, we shall often write the equations for the (12) pair only, understanding the two other equations obtained by circular permutations on the ijk indexes). Solving

(51) with respect to $\psi_{12}(p_{30})$ gives a wave function which is analytical in the $\mathrm{Im}(p_{30})<0$ half plane, and confirms thus the existence of such solutions. Furthermore, equation (51) shows that the three projections $\Lambda_k^+\psi_{ij}(E_k)$ are equal (let us call them ψ) and satisfy the 3D equation

$$\psi = \frac{\Lambda^{+++}}{P_0 - E_1 - E_2 - E_3}\ \left[\, V_{12}^0 + V_{23}^0 + V_{31}^0 \,\right]\ \psi. \tag{52}$$

Moreover, ψ is the integral of the Bethe-Salpeter amplitude with respect to the relative times:

$$\psi = \frac{-1}{2i\pi}\int dp_0\,\Phi \equiv \frac{-1}{2i\pi}\int dp_{10}dp_{20}dp_{30}\,\delta(p_{10}+p_{20}+p_{30}-P_0)\,\Phi. \tag{53}$$

Let us now choose

$$K_{ij}^0\ =\ K_{ij}^{T++}(\,s_{ij},\,s_{ij},\,P_0-h_k\,) \tag{54}$$

The 3D equation (52) is then built with the two-body potentials $V_{ij}^0{=}V_{ij}(P_0{-}h_k)$, with the V_{ij} defined by (29) in section 2 for each fermion pair, and becomes equation (45).

Let us now consider a non-approximated three-fermion Bethe-Salpeter equation (46) and try to transform it into our 3D equation (52) above, completed by higher-order correction terms. It first seemed us a good idea to start with approximations giving the 3D equation (52) in a more direct way than above, like

$$G_1^0G_2^0\ \approx\ G_{12}^{\delta P} \tag{55}$$

$$G_1^0G_2^0\,\delta(p'_{30}-p_{30})\ \approx\ G_{12}^{\delta P}\,\delta(p'_{30}-h_3) \tag{56}$$

$$K_{12}\,\delta(p'_{30}-p_{30})\ \approx\ \Lambda_3^+\,K_{12}^0\,\delta(p'_{30}-E_3). \tag{57}$$

Each of these approximations gives in each case the 3D equation (52), but the correction terms spoil the result: with (55) or (56) they contain supplementary zero-order contributions. With (57) they contain no supplementary zero-order contributions, but the correction terms can not easily be sorted by increasing order. We must thus proceed more carefully. We finally decided to start with positive-energy, instantaneous and cluster-energy independent approximations of the two-body kernels, like (54) or its Born approximation, adding the third Λ_k^+ projector in order to avoid unnecessary complications, and to proceed in three steps, inspired by the demonstration (46)-(52):

i) Transform the Bethe-Salpeter equation (46) into a set of three coupled equations for the $\psi_{ij}(p_{k0})$.

ii) Transform this set into a set of three coupled 3D equations for the $\psi_{ij}(E_k)$ (these are *not* the Faddeev equations).

iii) Transform this last set into a single 3D equation for ψ, defined as the mean of the $\psi_{ij}(E_k)$.

3.3 Three coupled equations for three cluster-energy depending amplitudes.

Let us now consider the unapproximated Bethe-Salpeter equation (46) and define

$$G^0\ =\ G_1^0\,G_2^0\,G_3^0\,, \tag{58}$$

$$K\ =\ (G_3^0)^{-1}K_{12}\ +\ (G_1^0)^{-1}K_{23}\ +\ (G_2^0)^{-1}K_{31}\ +\ K_{123}. \tag{59}$$

The Bethe-Salpeter equation (46) takes then the compact form:

$$\Phi\ =\ G^0K\,\Phi. \tag{60}$$

Let us also define

$$K^0\ =\ (G_3^0)^{-1}K_{12}^0\ +\ (G_1^0)^{-1}K_{23}^0\ +\ (G_2^0)^{-1}K_{31}^0\,, \tag{61}$$

$$K^R\ =\ K-K^0\,, \tag{62}$$

each K_{ij}^0 being independent of all relative energies and containing Λ^{+++} projectors (in order to render the following calculations simpler, we now add the Λ_k^+ projectors to the Λ_{ij}^{++} projectors of the previous section). Let us now perform the transformations

$$\Phi = G^0K^0\Phi + G^0K^R\Phi \tag{63}$$

$$\Phi = (1-G^0K^R)^{-1}G^0K^0\Phi = G^K\,K^0\Phi \tag{64}$$

where

$$G^K \;=\; G^0 + G^0 K^R G^0 + G^0 K^R G^0 K^R G^0 + \cdots \equiv\; G^0 + G^{KR}. \tag{65}$$

Since eq. (64) ends with $K^0\Phi$, we can transform it into a set of three coupled equations for the $\psi_{ij}(p_{k0})$, now defined with the Λ^{+++} projector:

$$\psi_{ij}(p_{k0}) = \Lambda^{+++}\,\beta_k\,G^{-1}_{0k}\int dp_{ij0}\,\Phi. \tag{66}$$

We get

$$\psi_{12}(p'_{30}) \;=\; \sum_k \int dp_{k0}\,G^K_{12,ij}(p'_{30},p_{k0})\,K^0_{ij}\,\psi_{ij}(p_{k0}) \tag{67}$$

with

$$G^K_{12,ij}(p'_{30},p_{k0}) \;=\; \Lambda^{+++}\,\beta_3\left[G^0_3(p'_{30})\right]^{-1}\int dp'_{120}\,dp_{ij0}\,G^K(p'_0,p_0)\;\beta_k \tag{68}$$

3.4 Three coupled 3D equations.

Let us now compute the contribution of G^0 to (68). We get (omitting the Λ^{+++} already contained in K^0):

$$G^0_{12,12}(p'_{30},p_{30}) \;=\; \frac{-2i\pi}{(P_0-S)-(p'_{30}-E_3)+i\epsilon}\,\beta_1\beta_2\,\delta(p'_{30}-p_{30}) \tag{69}$$

$$G^0_{12,23}(p'_{30},p_{10}) \;=\; \beta_3\,G^0_1(p_{10})\,G^0_2(P_0-p_{10}-p'_{30})\,\beta_1 \tag{70}$$

$$G^0_{12,31}(p'_{30},p_{20}) \;=\; \beta_3\,G^0_2(p_{20})\,G^0_1(P_0-p_{20}-p'_{30})\,\beta_2 \tag{71}$$

The product of propagators (70) will be included in a clockwise path integral in p_{10}. With positive-energy instantaneous kernels, this integral is given by the residue of the pole of $G^0_1(p_{10})$ at $p_{10}=E_1$. This is no more true in the general case, but we can still isolate a part of $G^0_1(p_{10})$ which leads to the same result:

$$G^0_1(p_{10}) \;=\; G^\delta_1(p_{10}) \;+\; G^R_1(p_{10}) \tag{72}$$

with

$$G^\delta_1(p_{10}) \;=\; -2i\pi\,\delta(p_{10}-E_1)\,\beta_1 \tag{73}$$

$$G^R_1(p_{10}) \;=\; \frac{1}{p_{10}-E_1-i\epsilon}\;\beta_1 \tag{74}$$

so that

$$G^\delta_{12,23}(p'_{30},p_{10}) \;=\; \frac{-2i\pi}{(P_0-S)-(p'_{30}-E_3)+i\epsilon}\,\beta_2\beta_3\,\delta(p_{10}-E_1) \tag{75}$$

and similably

$$G^\delta_{12,31}(p'_{30},p_{20}) \;=\; \frac{-2i\pi}{(P_0-S)-(p'_{30}-E_3)+i\epsilon}\,\beta_3\beta_1\,\delta(p_{20}-E_2). \tag{76}$$

Putting all together, we have finally

$$\begin{aligned}
\psi_{12}(p'_{30}) \;=\; & \frac{-2i\pi}{(P_0-S)-(p'_{30}-E_3)+i\epsilon}\,\beta_1\beta_2\,K^0_{12}\,\psi_{12}(p'_{30}) \\
& + \frac{-2i\pi}{(P_0-S)-(p'_{30}-E_3)+i\epsilon}\left[\,\beta_2\beta_3\,K^0_{23}\,\psi_{23}(E_1) \;+\; \beta_3\beta_1\,K^0_{31}\,\psi_{31}(E_2)\,\right] \\
& + \int dp_{10}\,\beta_3\,G^R_1(p_{10})\,G^0_2(P_0-p_{10}-p'_{30})\,\beta_1\,K^0_{23}\,\psi_{23}(p_{10}) \\
& + \int dp_{20}\,\beta_3\,G^R_2(p_{20})\,G^0_1(P_0-p_{20}-p'_{30})\,\beta_2\,K^0_{31}\,\psi_{31}(p_{20}) \\
& + \sum_k \int dp_{k0}\,G^{KR}_{12,ij}(p'_{30},p_{k0})\,K^0_{ij}\,\psi_{ij}(p_{k0})
\end{aligned} \tag{77}$$

If we neglect the last term of (77) the two preceding terms vanish also by integration, and we recover the result of the positive-energy instantaneous approximation. We shall thus move the first term of (77) to

the left-hand side, consider the two next terms (second line) as the principal contributions, and the three last terms as perturbations. The difference between this approach and that based on (57) lies in the fact that we do not replace immediately p'_{30} by E_3 in the first term. We get

$$\psi_{12}(p'_{30}) \,=\, g_{12}(p'_{30})\,\big[\,V^0_{23}\,\psi_{23}(E_1) \,+\, V^0_{31}\,\psi_{31}(E_2)\,\big]$$

$$+\, g_{12}(p'_{30})\,[\,g^0_{12}(p'_{30})\,]^{-1}\,\sum_k \int dp_{k0}\,G^{KT}_{12,ij}(p'_{30},p_{k0})\,K^0_{ij}\,\psi_{ij}(p_{k0}) \tag{78}$$

where $G^{KT}_{12,ij}$ is defined from $G^{KR}_{12,ij}$ by including the 3th and the 4th lines of (77) in the last one, and

$$g_{12}(p'_{30}) \,=\, \frac{1}{(P_0 - S - V^0_{12}) - (p'_{30} - E_3) + i\epsilon} \tag{79}$$

$$g^0_{12}(p'_{30}) \,=\, \frac{1}{(P_0 - S) - (p'_{30} - E_3) + i\epsilon} \tag{80}$$

The iterations of (78) lead to

$$\psi_{12}(p'_{30}) \,=\, g_{12}(p'_{30})\,\big[\,V^0_{23}\,\psi_{23}(E_1) \,+\, V^0_{31}\,\psi_{31}(E_2)\,\big]$$

$$+\, g_{12}(p'_{30})\,[\,g^0_{12}(p'_{30})\,]^{-1}\,\sum_k \int dp_{k0}\,G^{TT}_{12,ij}(p'_{30},p_{k0})\,K^0_{ij}$$

$$g_{ij}(p_{k0})\,\big[\,V^0_{jk}\,\psi_{jk}(E_i) \,+\, V^0_{ki}\,\psi_{ki}(E_j)\,\big]. \tag{81}$$

defining G^{TT} as the sum of the iterations of G^{KT} :

$$G^{TT}_{12,ij}(p'_{30},p_{k0}) \,=\, G^{KT}_{12,ij}(p'_{30},p_{k0}) \,+\, \sum_{k'} \int dp''_{k'0}\,G^{KT}_{12,i'j'}(p'_{30},p''_{k'0})\,K^0_{i'j'}$$

$$g_{i'j'}(p''_{k'0})\,[\,g^0_{i'j'}(p''_{k'0})\,]^{-1} G^{KT}_{i'j',ij}(p''_{k'0},p_{k0}) \,+\cdots \tag{82}$$

We can now take (81) at $p'_{30}=E_3$, multiply at left by $[\,g_{12}(E_3)]^{-1}$, and move the term in $V^0_{12}\psi_{12}(E_3)$ back to the right-hand side:

$$\psi_{12}(E_3) \,=\, \frac{1}{P_0 - S + i\epsilon}\,\big[\,V^0_{12}\,\psi_{12}(E_3) \,+\, V^0_{23}\,\psi_{23}(E_1) \,+\, V^0_{31}\,\psi_{31}(E_2)\,\big]$$

$$+\,\sum_k M_{12,ij}\,\big[\,V^0_{jk}\,\psi_{jk}(E_i) \,+\, V^0_{ki}\,\psi_{ki}(E_j)\,\big] \tag{83}$$

with

$$M_{12,ij} \,=\, \int dp_{k0}\,G^{TT}_{12,ij}(E_3,p_{k0})\,K^0_{ij}\,g_{ij}(p_{k0}). \tag{84}$$

3.5 One 3D equation.

Let us write (83) and the two other equations obtained by circular permutations in matricial form. We shall define

$$\Psi^k \,=\, \psi_{ij}(E_k), \qquad \Pi^{ij} \,=\, \frac{1}{3}, \quad \overline{\Pi} \,=\, 1 - \Pi. \tag{85}$$

The matrix Π is the projector ($\Pi^2 = \Pi$) on the $u^k = 1$ vector (of norm 3). We have

$$\Pi\,\Psi \,=\, \psi\,u\,, \qquad \psi \,=\, \frac{1}{3}\,(\,\Psi^1 \,+\, \Psi^2 \,+\, \Psi^3\,). \tag{86}$$

We can write eq. (83) in the form

$$\Psi \,=\, M\,\Psi, \quad M = M^0 + M^R, \quad M^{0i1} \,=\, \frac{1}{P_0 - S + i\epsilon}\,V^0_{23},\ etc... \tag{87}$$

The action of the projector Π on M^0 is

$$\Pi\, M^0 \,=\, M^0, \quad M^0\,\Pi \,=\, \frac{\Pi}{P_0 - S + i\epsilon}\, V^0, \quad V = V_{12}^0 + V_{23}^0 + V_{31}^0. \tag{88}$$

We shall now transform the matricial equation (87) into a scalar equation for ψ. The expansion of (87) with respect to $\Pi\Psi$ gives

$$\Pi\,\Psi \,=\, \Pi\, M\,(\,1 + \overline{\Pi} M + \overline{\Pi} M\,\overline{\Pi} M + \cdots)\,\Pi\,\Psi \tag{89}$$

in which $\overline{\Pi} M = \overline{\Pi} M^R$. Taking the scalar product with u leads finally to

$$\psi \,=\, \left[\, \frac{1}{P_0 - S + i\epsilon}\, V^0 \,+\, \frac{1}{3}\, u^\top\,(\,M^R + M\,[\,\overline{\Pi} M^R + \overline{\Pi} M^R\,\overline{\Pi} M^R + \cdots]\,)\,u \,\right]\,\psi. \tag{90}$$

In this equation, V^0 contains the projector Λ^{+++} at left and at right, while M^R contains it at right. We can thus trivially transform (90) into an equation for $\Lambda^{+++}\psi$.

3.6 First-order energy shift.

The 3D reduction above is rather complicated, since it implies three levels of series expansions. The computation of the first-order energy shift remains however tractable. The energy shift is, keeping only the terms which contribute at first-order:

$$P_0 - P_0^{(0)} \,=<\,(P_0 - S)\,\frac{1}{3}\,u^\top\,(\,1 + M^0\overline{\Pi}\,)\,M^R\,u\,>\,. \tag{91}$$

We can make the replacements

$$1 + M^0\overline{\Pi} \,=\, 1 + M^0 - \frac{\Pi}{P_0 - S}\,V^0 \,\rightarrow\, M^0 \tag{92}$$

as the contributions of the first and third terms cancel mutually. We remain with

$$\begin{aligned} P_0 - P_0^{(0)} \,&=<\,(P_0 - S)\,\frac{1}{3}\,u^\top\,M^0\,M^R\,u\,> \\ &= \sum_{k',k} <\,V_{i'j'}^0\,M_{i'j',ij}\,(\,V^0 - V_{ij}^0\,)\,> \end{aligned} \tag{93}$$

The contribution of the terms with $i'j' = 12$ is then

$$\Delta_{12} \,=<\,V_{12}^0 \sum_k \int dp_{k0}\, G_{12,ij}^{TT}(E_3, p_{k0})\, K_{ij}^0\, g_{ij}(p_{k0})\,(\,V^0 - V_{ij}^0\,)\,> \tag{94}$$

At first order, we can replace $G_{12,ij}^{TT}$ by the non-iterated $G_{12,ij}^{KT}$ and further by $G_{12,ij}^{KR}$, as the 3th and 4th lines of (77) will lead to null integrals. Using the definitions (65), (68) we get

$$\begin{aligned} \Delta_{12} \,=<\,V_{12}^0 \sum_k \int \delta(p'_{30} - E_3) dp'_0\, dp_0\, \beta_3\,\left[G_3^0(p'_{30})\right]^{-1} \\ G^{KR}(p'_0, p_0)\;\beta_k\, K_{ij}^0\, g_{ij}(p_{k0})\,(\,V - V_{ij}^0\,)\,> \end{aligned} \tag{95}$$

Replacing V by $P_0 - S$ and G^{KR} by $G^0 K^R G^0$ leads to

$$P_0 - P_0^{(0)} \,=\, \frac{-1}{4\pi^2} <\, \int dp'_0\, dp_0\, V_F(p'_0)\, G^0(p'_0) K^R(p'_0, p_0) G^0(p_0)\, \beta_1\beta_2\beta_3\, V_I(p_0)\,> \tag{96}$$

with

$$V_I(p_0) \,=\, \sum_k V_{ij}^0\, g_{ij}(p_{k0})\, g_{ij}^{-1}(E_k) \tag{97}$$

$$V_F(p'_0) \;=\; -2i\pi \sum_{k'} V^0_{i'j'}\,\delta(p'_{k'0}-E'_{k'})\,\beta_{k'}\,\left[G^0_{k'}(p'_{k'0})\right]^{-1}. \tag{98}$$

In writing (98), we mean that $\left[G^0_{k'}(p'_{k'0})\right]^{-1}$ cancels the $G^0_{k'}(p'_{k'0})$ contained in $G^0(p'_0)$ before the replacement of $p'_{k'0}$ by $E'_{k'}$. The energy shift (96) contains 36 terms: there are 4 terms in K^R (if we do not forget the irreducible three-body term) times 3 terms in V_I and 3 terms in V_F. Our reason for introducing these V_I and V_F is the fact that they both could be replaced by V^0 if K^R were instantaneous.

The singularities to consider when performing the integrations are the poles of the propagators, the singularities of K^R and the poles of V_I. It is always possible to choose the integration paths, in each term, in order to avoid contributions from the poles of V_I. The contributions of the poles of the propagators to the energy shift (96) are then easy to compute, as $g_{ij}(p_{k0})$ in (97) is to be taken at $p_{k0}=h_k$. We get

$$R_{12} \;=<\sum_{k'k} V^0_{i'j'}\,\frac{1}{P_0-S}\,V^{Rba}_{12}\,\frac{1}{P_0-S}\,V^0_{ij}> \tag{99}$$

plus the similar R_{23} and R_{31} and a contribution R_{123} of the irreducible three-body kernel. By V^{Rba}_{12} we denote $-2i\pi\beta_1\beta_2 K^R_{12}$ with the initial fermion a, the final fermion b and the spectator fermion 3 on their positive-energy mass shells. We have to consider 9 terms in 4 different groups:

$k'=k=3$	$ba=11,\ 12,\ 21\ \text{or}\ 22$
$k'=1\,\text{or}\,2,\ k=3$	$ba=k'1\ \text{or}\ k'2$
$k'=3,\ k=1\,\text{or}\,2$	$ba=1\,k\ \text{or}\ 2\,k$
$k'=1\,\text{or}\,2\,,\ k=1\,\text{or}\,2$	$ba=k'k$

We still have some freedom in the choice of ba which we could use to get a real energy shift, for example by choosing $a=b$ in the three first groups. The contributions of an instantaneous part K^{RI}_{12} of K^R_{12} would sum into a $< V^{RI}_{12} >$ but we could also include this K^{RI}_{12} directly in K^0_{12}. The corrections corresponding to the differences between the various V^{Rba}_{12} are a step towards Gross' spectator model (see below, section 6 and [17, 18]).

The principal task remains however the calculation of the contributions of the singularities of the K^R_{ij}, for the integration paths defined by our choices of ab above. As we shall see in section 5.3, these contributions could be as important as these coming directly from the relative time dependence of these kernels (they could however be partially cancelled by the contributions of two-body non-ladder terms and three-body terms [8, 12, 17]). A general expression of the contributions of these singularities can be obtained by splitting the propagators containing the poles considered in (99), using (72), the all-G^δ part giving (99).

For more explicit results, we should specialize to specific three-fermion problems. It must be noted that the ratio $g_{ij}(p_{k0})\,g^{-1}_{ij}(E_k)$ of (97) is no more 1 in the terms containing $G^R_k(p_{k0})$. If the singularities of K^R in the complex p_{k0} plane lie far from the positive-energy mass shell, this ratio can be approximated by $-(p_{k0}-E_k)^{-1}\,g^{-1}_{ij}(E_k)$ and takes a relatively low value. This suggests us to neglect the contributions of the terms containing $G^R_k(p_{k0})$ to (96), as these contributions contain at least one large denominator more than the contributions of the terms containing $G^\delta_k(p_{k0})$. This leads to the replacement

$$V_I(p_0) \to = -2i\pi \sum_{k\neq u} V^0_{ij}\,\beta_k\,\left[G^0_k(p_{k0})\right]^{-1}\delta(p_{k0}-E_k) \;+\; \sum_{k=u} V^0_{ij} \tag{100}$$

in (96). We isolated the contributions of the unconnected terms $k=u$ (like $k=3$ combined with $k'=3$ and K^R_{12}). The resulting energy shift is now manifestly real.

3.7 Cluster separability.

Physically, if we "switch off" the (23), (31) and (123) interactions, we remain with a pair of interacting fermions (12) and a free fermion (3). We would like to read this on our final 3D equation. however:

– The cluster-separability condition may be satisfied by an exact 3D reduction, but not by its truncated forms.

– Even with an exact 3D reduction, the cluster separability property is a welcome bonus, not a requirement: the Bethe-Salpeter equation (46), from which we started, is indeed correct for three-fermion bound states only.

In our calculations above, the cluster-separability property is in general spoiled by our iteration of (78), which starts with a term which vanishes at the (12) cluster-separated limit. The (12)-limit equation is then built with the approached $-2i\pi\beta_1\beta_2 K_{12}^0$ only. The cluster separability can however be restored by performing a supplementary iteration, like our replacement of V^0 by $P_0 - S$ in (93), leading to the cluster-separable correction (99). Another possibility consists in our previous choice (54) of the K_{ij}^0. With this choice, the cluster-separated limits are exact. Moreover, the corresponding two-body scattering amplitudes are also correct (of course, we do not mean that we should keep all the terms of the K_{ij}^0 in an actual computation: we should only keep the ones which contribute up to the desired order). Simpler choices (as the Born approximations) would however not be uncorrect in the calculation of the three-fermion bound state spectrum.

4 Other propagator-based 3D reductions.

4.1 Two fermions.

We have already seen two different choices of G^δ : Salpeter's propagator $G^{\delta S}$ (19) and the positive-energy propagator $G^{\delta P}$ (20), which we used through this work until now. We could also use Breit's propagator [39]:

$$G^{\delta B}(p_0) = -2i\pi\,\delta(p_0 - s)\,g^0\,\beta_1\beta_2. \tag{101}$$

in the two-fermion problem only, as it would lead to continuum dissolution in the two-fermion in an external potential and in the three-fermion problems. The first-order constraint $\delta(p_0 - s)$ could also be replaced by the second-order constraint $\delta(p_0 - \mu)$, which leads to a simpler one-body limit, especially with Breit's propagator (see section 7 below). In the spirit of Gross'spectator model, we could also choose the unsymmetrical or symmetrized propagators

$$G^{\delta G1}(p_0) = -2i\pi\,\Lambda_2^+\delta(p_{20} - E_2)\,g^0\,\beta_1\beta_2. \tag{102}$$

$$G^{\delta GS}(p_0) = -2i\pi\,\frac{1}{2}\,\left[\Lambda_2^+\delta(p_{20} - E_2) + \Lambda_1^+\delta(p_{10} - E_1)\right]\,g^0\,\beta_1\beta_2. \tag{103}$$

in the two-fermion problem (for the three-fermion problem, see below section 6).

With Sazdjian's covariant propagator [6, 12, 19]

$$\begin{aligned} G^{SZ} \;&=\; -2i\pi\,\frac{(p_1\cdot\gamma_1 + m_1)\,(p_2\cdot\gamma_2 + m_2)}{p_1^2 + p_2^2 - (m_1^2 + m_2^2) + i\epsilon}\;\delta\,(P\cdot p - \frac{m_1^2 - m_2^2}{2}\,) \\ &= -2i\pi\,\frac{P_0 + S}{2|P_0|}\,\delta(p_0 - \mu)\,g^0\,\beta_1\beta_2. \end{aligned} \tag{104}$$

the 3D potential is given by a series of separately covariant terms, so that we can truncate it without breaking the covariance. In order to preserve this property in the two-fermion in an external potential problem and in the three-fermion problem, without meeting continuum dissolution, we must however combine this propagator with a covariant substitute of the noncovariant Λ^{++}. A possible choice is the product

$$\theta^{++} = \theta(p_{10})\theta(p_1^2)\theta(p_{20})\theta(p_2^2) = \theta(p_{10} - |\,\vec{p}_1|\,)\,\theta(p_{20} - |\,\vec{p}_2|\,) \tag{105}$$

in which the constraint will replace p_{i0} by $\frac{1}{2}P_0 \pm \mu$. The sign of p_{i0} is indeed invariant when $p_i^2 > 0$. Another difficulty comes from the $(P_0 + S)/2|P_0|$ operator which obliges us to perform supplementary manipulations after the 3D reduction in order to render the potential hermitian, so that the Born term itself is finally given by a series. We examined elsewhere [40] how to use Sazdjian's propagator in the two-fermion, two fermions in an external potential and three-fermion problems, leaving the reader decide wether the covariance of the approximations balances the supplementary complications.

With Salpeter's propagator, the 3D potential will be

$$V^S = -2i\pi\,\tau\,\beta_1\beta_2\,K^{TS}(s,s)\,\tau^2 \tag{106}$$

as $\tau^2\psi^S = \psi^S$, with K^{TS} built with $G^{\delta S}$ like K^{TP} with $G^{\delta P}$. This potential is hermitian in the $\tau^2 = 1$ subspace at fixed P with the scalar product

$$(\psi_i, \psi_j) \;=\; \int d^3p\;\psi_i^+(\vec{p})\,\tau(\vec{p})\,\psi_j(\vec{p}). \tag{107}$$

4.2 Three fermions and continuum dissolution.

The results of a 3D reduction of the two-fermion Bethe-Salpeter equation can be used in the three-fermion problem at two levels: by choosing the corresponding approximations of the two-fermion kernels in the Bethe-Salpeter equation (46) , or by combining directly the corresponding potentials into a 3D equation (42). Both approaches turn out to be equivalent in the case of our 3D reduction based on $G^{\delta P}$. With the other choices, the 3D equation obtained by reducing the approximated Bethe-Salpeter equation would contain supplementary terms. In all cases, we get an exact three-fermion equation by "switching off" the interactions with the third fermion. However, the three-fermion 3D equation built with the potentials V^B or V^{SZ}, based on approximated propagators without projectors, suffers of continuum dissolution. It is indeed possible to build a continuum of solutions with any a priori given total energy by combining asymptotically free fermions with opposite energy signs. Any physical bound state is thus mixed with such a continuum and the building of normalisable bound state wave functions becomes impossible [13-16]. In the pure two-body case the energy of a system $(+,-)$ in the total rest frame is

$$E_1 - E_2 = \sqrt{\vec{p}^2 + m_1^2} - \sqrt{\vec{p}^2 + m_2^2} = \frac{m_1^2 - m_2^2}{\sqrt{\vec{p}^2 + m_1^2} + \sqrt{\vec{p}^2 + m_2^2}} \tag{108}$$

and lies thus between $m_1 - m_2$ (whichever the sign) and zero. We have thus no problem if we make the assumption that the energies of the bound states lie between $|\, m_1 - m_2|$ and $m_1 + m_2$. Below $|\, m_1 - m_2|$ we meet the "strong field" problem. The strong field problem for the one-body plus potential and two-body systems, the continuum dissolution problem for the two-body plus potential and three-body systems are both consequences of the possibility of pair creation.

In the three-body case, the mixing with asymptotically separated (12)(3) subsystems of opposite energy signs is excluded by the total momentum conservation, as in the two-fermion case. For the mixing with three-fermion asymptotically free states, let us consider for example

$$E_3 - E_1 - E_2 = \sqrt{\vec{p}_3^{\,2} + m_3^2} - \sqrt{\vec{p}_1^{\,2} + m_1^2} - \sqrt{\vec{p}_2^{\,2} + m_2^2}\,, \qquad \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \vec{0}. \tag{109}$$

There is no lowest value (we can for example have $\vec{p}_3 = 0$ and $\vec{p}_1 = -\vec{p}_2$ arbitrarily large). The highest values are obtained when $\vec{p}_1$ and $\vec{p}_2$ have the same direction, and for $\vec{p}_1/m_1 = \vec{p}_2/m_2 = -\vec{p}_3/(m_1+m_2)$. In this case we have

$$E_3 - E_1 - E_2 = \sqrt{\vec{p}_3^{\,2} + m_3^2} - \sqrt{\vec{p}_3^{\,2} + (m_1 + m_2)^2} \tag{110}$$

so that $E_3 - E_1 - E_2$ lies finally between $-\infty$ and $(m_3 - m_1 - m_2)\theta(m_3 - m_1 - m_2)$. Symmetrically, $E_1 + E_2 - E_3$ lies between $(m_1+m_2-m_3)\theta(m_3-m_1-m_2)$ and ∞. If we assume that the energies of the bound states lie between $(m_1+m_2+m_3-2\,\mathrm{Inf}(m_i))$ (i.e. above the highest negative-energy threshold) and $(m_1+m_2+m_3)$, there is no degenerescence with the "one plus-two minus" states. On the contrary, no weak field assumption could prevent the degenerescence with the "two plus-one minus" states.

The continuum dissolution problem is avoided with the potentials containing the positive-energy projectors Λ_{ij}^{++}, or their covariant substitutes θ_{ij}^{++}. The 3D equation built with Salpeter's potentials V_{ij}^S, which include a τ_{ij} at left and a τ_{ij}^2 at right, can be splitted into two uncoupled equations in the $\tau_1\tau_2\tau_3 = \pm 1$ subspaces and the solutions are eigenstates of $\tau_1\tau_2\tau_3$ and free of continuum dissolution. The $\tau_1\tau_2\tau_3 = +1$ subspace includes the three-plus and the two-minus one-plus components. The interaction term is hermitian for a scalar product built with $\tau_{123} = \mathrm{sign}\,(\tau_1 + \tau_2 + \tau_3)$. Equation (42) becomes then a logical three-fermion extension of Salpeter's equation and exhibits a particle-antiparticle symmetry. We do not expect however to get a large contribution from the two-minus one-plus components.

5 Kernel-based 3D reductions.

5.1 Two fermions.

Our 3D reduction of the two-fermion Bethe-Salpeter equation was built around an approximation of the free propagator. However, the first step of our 3D reduction of the three-fermion Bethe-Salpeter equation is built around instantaneous approximations of the kernels, while the second step is again built around approximations of the free propagators. It is therefore interesting to examine also how a 3D reduction

of the two-fermion Bethe-Salpeter equation can be built around an instantaneous approximation of the kernel. Let us thus consider an approximation K^Z of the Bethe-Salpeter kernel:

$$K \,=\, K^Z \,+\, K^R. \tag{111}$$

The Bethe-Salpeter equation becomes

$$\Phi = G^0 K^Z \Phi + G^0 K^R \Phi \tag{112}$$

$$\Phi = (1 - G^0 K^R)^{-1} G^0 K^Z \Phi = G^K K^Z \Phi \tag{113}$$

with

$$G^K \,=\, G^0 \,+\, G^0 K^R G^0 + G^0 K^R G^0 K^R G^0 + \cdots \equiv\, G^0 + G^{KR}\,. \tag{114}$$

If we now specialize K^Z to an instantaneous positive-energy kernel (K^Z independent of p_0 and equal to $\beta_1\beta_2\Lambda^{++}\beta_1\beta_2 K^Z \Lambda^{++}$), eq. (113) leads to the 3D equation

$$\phi \,=\, (g^0 + g^{KR})\, V^Z\, \phi \tag{115}$$

with

$$\phi = \Lambda^{++} \int dp_0\, \Phi(p_0), \qquad V^Z = -2i\pi\, \beta_1\beta_2 K^Z, \tag{116}$$

$$g^{KR} = -\frac{1}{2i\pi}\, \Lambda^{++} \int dp_0'\, dp_0\, G^{KR}(p_0', p_0)\, \beta_1\beta_2\, \Lambda^{++}. \tag{117}$$

The first-order energy shift is, with the replacement $g^{KR}V^Z\phi \approx g^{KR}(P_0 - S)\phi$:

$$P_0 - P_0^{(0)} = -\frac{1}{2i\pi} \,<\, (P^0 - S) \int dp_0'\, dp_0\, G^0(p_0') K^R(p_0', p_0) G^0(p_0)\, \beta_1\beta_2\, (P^0 - S) \,>\,. \tag{118}$$

Here again, all elements of (118) are total-energy dependent. Although (118) is symmetric, the perturbation potential $(P_0 - S)g^{KR}V^Z$ of (115) is not. This does not exclude the possibility of a real energy spectrum: if we admit an energy dependence of the potential (which we cannot avoid), we have also the possibility of performing an infinity of rearrangements of the equation. We can for example render the potential symmetric by treating g^{KR} as a perturbation of g^0 (like G^R as a perturbation of G^δ in section (2.2)). We can equivalently start with the corresponding 4D equation (113):

$$\Phi = (G^0 + G^{KR})\, K^Z \Phi = \phi + G^{KR}\, K^Z \Phi \tag{119}$$

$$\phi = G^0 K^Z \Phi \,= G^0 K^Z (1 - G^{KR}\, K^Z)^{-1}\, \phi = G^0 K^K \phi \tag{120}$$

where

$$K^K \,= K^Z \,+\, K^Z G^{KR} K^Z \,+\, \cdots \tag{121}$$

is a symmetric positive-energy instantaneous potential, since it begins and ends with K^Z. The 3D equation is then

$$\eta \,=\, g^0\, V^K\, \eta, \qquad V^K \,=\, -2i\pi\, \beta_1\beta_2\, K^K, \qquad \eta = \int dp_0\, \phi\,(p_0) \tag{122}$$

and the first-order energy shift is still given by (118). As for the propagator-based reduction, we can represent the various contributions by Feynman graphs. We can indeed get K^K by considering the transition operator

$$T \,=\, (K^Z + K^R) \,+\, (K^Z + K^R)\, G^0\, (K^Z + K^R) \,+\, \cdots \tag{123}$$

and skipping some terms according to the two rules:

– Each term must begin and end with K^Z

– There must be at least one K^R between two consecutive K^Z.

The 3D transition operator is now $-2i\pi\beta_1\beta_2 T^K$ with

$$T^K = K^K + K^K G^0 K^K + \cdots = K^K (1 - G^0 K^K)^{-1}$$

$$= K^Z (1 - G^{KR} K^Z)^{-1} (1 - G^0 K^Z (1 - G^{KR} K^Z)^{-1})^{-1}$$

$$= K^Z (1 - G^{KR} K^Z - G^0 K^Z)^{-1} = K^Z (1 - G^K K^Z)^{-1}$$

$$= K^Z(1-(1-G^0K^R)^{-1}G^0K^Z)^{-1}$$

$$= K^Z(1-G^0K^R-G^0K^Z)^{-1}(1-G^0K^R) = K^Z(1-G^0K)^{-1}(1-G^0K^R)$$

$$K^Z(1+G^0T)(1-G^0K^R) \tag{124}$$

Replacing then K^R by $K-K^Z$ and $T(1-G^0K)$ by K gives

$$T^K = K^Z + K^ZG^0K^Z + K^ZG^0TG^0K^Z. \tag{125}$$

In contrast with what happens in the propagator-based reductions, the on mass shell restriction of this operator is not the physical scattering amplitude of field theory. In fact, the Bethe-Salpeter equation (1), without inhomogeneous term, is a bound state equation, and for the bound states we are only interested in the poles of (125). These poles are the poles of T, unless the integrations on p'_0, p_0 make some residues vanish. Getting also the correct physical scattering amplitudes was a welcome bonus of the propagator-based reductions.

If we want T in terms of T^K, we can write (124) as

$$T^K(1-G^0K^R)^{-1} = K^Z(1+G^0T) = (K-K^R)(1+G^0T)$$

$$= T - K^R(1+G^0T) = (1-K^RG^0)T - K^R \tag{126}$$

$$T = (1-K^RG^0)^{-1}T^K(1-G^0K^R)^{-1} + (1-K^RG^0)^{-1}K^R$$

$$= (1+T^RG^0)T^K(1+G^0T^R) + T^R \tag{127}$$

with

$$T^R = K^R + K^RG^0K^R + \cdots. \tag{128}$$

We have a large freedom in the choice of K^Z. We could choose

$$K^Z = \beta_1\beta_2\,\Lambda^{++}\,\beta_1\beta_2\,K(s,s)\,\Lambda^{++}. \tag{129}$$

We could also keep only the ladder term of K, or a part of it in (129), like the Coulomb part of the one-photon exchange contribution. We could use K^T (in this case we expect that the remainder of K^K does not finally contribute to the total energy spectrum) or an approximation of it (one and two photon exchange terms, for example). Phillips and Wallace [20] suggested to choose K^Z in order to annihilate g^{KR} exactly or up to a given order. If we choose to annihilate it at first-order, the first-order energy shift (118) vanishes and we get

$$K^Z = \frac{-1}{(2\pi)^2}\,\beta_1\beta_2\,(g^0)^{-1}\Lambda^{++}\int dp'_0\,dp_0\,G^0(p'_0)K(p'_0,p_0)G^0(p_0)\,\beta_1\beta_2\,(g^0)^{-1}\Lambda^{++}. \tag{130}$$

The skipping of terms in (123) is then improved, as we must now have at last *two* K^R between two consecutive K^Z. If we choose to annihilate g^{KR} exactly, we get, using (127), (117) and the relation $G^{KR}=G^0T^RG^0$:

$$\frac{-1}{(2\pi)^2}\,\beta_1\beta_2\,(g^0)^{-1}\Lambda^{++}\int dp'_0\,dp_0\,G^0(p'_0)T(p'_0,p_0)G^0(p_0)\,\beta_1\beta_2\,(g^0)^{-1}\Lambda^{++} = T^K \tag{131}$$

which is the Λ^{++} projection of the expression given in [20] (they use Salpeter's propagator and write thus $\Lambda^{++}-\Lambda^{--}$ instead of Λ^{++}). In this case, T^K gives the correct physical scattering amplitudes when $P^0 \to E_1+E_2$, as the operators $(g^0)^{-1}$ kill the contributions of the singularities other than the poles of G^0.

5.2 Three fermions.

We could also use the results of the kernel-based 3D reduction of the two-fermion Bethe-Salpeter equation as the starting point of a 3D reduction of the three-fermion Bethe-Salpeter equation, by choosing

$$K^0_{ij} = K^K_{ij}(s,s,P_0-h_k) \tag{132}$$

as starting approximations in (54), in order to recover, at the 2+1 separated clusters limits, the kernel-based 3D equations.

5.3 One-photon exchange.

Let us now compute (130) with K given by the one-photon exchange graph, in Feynman gauge:

$$K \;=\; \frac{2ie^2}{(2\pi)^3}\;\frac{\gamma_1\cdot\gamma_2}{(p'-p)^2+i\epsilon}. \tag{133}$$

We have

$$K^Z \;=\; -\,\frac{2ie^2}{(2\pi)^5}\,\beta_1\beta_2\,\Lambda^{++}\,\gamma_1\cdot\gamma_2\,\Lambda^{++}\,I \tag{134}$$

With

$$\begin{aligned} I = \int dp'_{10}\,dp_{10}\quad & \frac{P_0-E'_1-E'_2}{(p'_{10}-E'_1+i\epsilon'_1)\,(P_0-p'_{10}-E'_2+i\epsilon'_2)} \\ \frac{1}{(p'_{10}-p_{10})^2-\vec{k}^2+i\eta}\quad & \frac{P_0-E_1-E_2}{(p_{10}-E_1+i\epsilon_1)\,(P_0-p_{10}-E_2+i\epsilon_2)} \end{aligned} \tag{135}$$

and $k=p'-p$. Closing the integration paths clockwise, and taking the $\epsilon_1\to 0$ limit before the $\eta\to 0$ limit in all terms, we get

$$I=-4\pi^2\;\frac{1}{(E'_1-E_1)^2-\vec{k}^2}\;\;(\,1\,+\,\frac{1}{2|\vec{k}|}\,R_K\,), \tag{136}$$

$$R_K \;=\; \frac{(E'_1-E_1-|\vec{k}|\,)(P_0-E_1-E_2)}{P_0-E'_1-E_2-|\vec{k}|}\;-\;\frac{(E'_1-E_1+|\vec{k}|\,)(P_0-E'_1-E'_2)}{P_0-E_1-E'_2-|\vec{k}|}. \tag{137}$$

The first term comes from the residues of the poles of the propagators only, while the terms of R_K combine the residues of one pole of the propagators and one pole of the kernel. Modifying the integration paths or the order of the limits would give the same result (after rearrangements). At lowest order, we get

$$R_K \;\approx\; 2P_0-E'_1-E'_2-E_1-E_2. \tag{138}$$

The contribution of R_K to the bound state energy will finally be in α^3, while the replacement of $(E'_1-E_1)^2-\vec{k}^2$ by simply $-\vec{k}^2$ (Coulomb potential) would lead to a correction in α^4. This means that the relative energy dependence of the kernel, at the ladder approximation, contributes mostly by providing supplementary poles inside the integration paths. Beyond the ladder approximation however, the contribution of the first crossed graph will cancel the leading term in α^3, leaving a contribution in α^4 [42]. This is not a surprise, as it is well known that, in well chosen propagator-based reductions, all higher-order contributions cancel mutually at the one-body limit [8, 12].

By rearranging (136, 137), we can also write

$$I \;=\; \frac{-4\pi^2}{2|\vec{k}|}\left(\frac{1}{P_0-E'_1-E_2-|\vec{k}|}\;+\;\frac{1}{P_0-E_1-E'_2-|\vec{k}|}\right) \tag{139}$$

which is the expression obtained in time-ordered perturbation theory [41]. Here, each term describes the emission of a photon by one fermion, followed by its absorption by the other one.

5.4 Time-ordered perturbation theory

In time-ordered perturbation theory (TOPT), one takes the Fourier transform of the full propagator (as given by the Feynman graphs method) with respect to the energy variables. Each graph becomes then splitted into several time-ordered graphs, and one integrates the result with respect to the time intervals. This leads to TOPT "Feynman rules". The TOPT graphs can be arranged in order to get an iterative equation for the propagator, and, by isolating the residues of the three-body bound state poles, to get an homogeneous 3D equation for the three-body bound state wave functions (this imitates the usual derivation of the inhomogeneous and homogeneous Bethe-Salpeter equations). The cluster-separability property is translated by the factorization of the propagator at the different vanishing interactions limits. The two and three-body 3D potentials are directly given, at any order, by the TOPT "Feynman rules". Up to now, this approach has been applied to a very simple model: the exchange of a scalar meson between two or three positive-energy bosons or fermions at the ladder approximation [41]. Our results in this work can be applied to more general interactions but should lead to more complicated calculations

than TOPT at higher-orders. A comparison of our first-order contributions computed with the simple Bethe-Salpeter kernel used in [41] should be interesting.

It must be noted that our motivation for introducing the positive-energy projectors is not that of [41]: we make the choice of approaching the exact equation for the positive-energy components, while [41] simply neglects these components as a temporary simplification. It will be interesting to see in which form the continuum dissolution problem appears in TOPT when the negative-energy components are taken into account.

6 Relation with Faddeev formalism and Gross' spectator model.

The nonrelativistic Faddeev equations can be obtained by transforming Schrödinger's equation (in a first step, without three-body terms) into a set of three coupled equations for three parts T_{ij} of the transition operator (T_{ij} denotes the contribution of all graphs beginning by a (ij) interaction). The input is the set of the three two-body transition operators and the resulting series expansion contains only connected graphs (never twice the same two-body transition operator). This formalism is well adapted to the description of the various scattering processes, such as $(12)+3 \to (12)+3$ (elastic scattering), $(12)+3 \to (12)^*+3$ (excitation), $(12)+3 \to 1+(23)$ (rearrangement), $(12)+3 \to 1+2+3$ (breakup). The (123) bound states correspond to the poles of this transition operator.

The structure of these equations can be generalized to relativistic equations which are not necessarily (exactly) reducible to a single 3D equation.

In Gross' spectator model [17, 18], Faddeev's type equations are deduced from the Bethe-Salpeter equation and the transition operator is written in terms of the two-body transition operators. The relative time variables are then eliminated by putting, in each three-body propagator, all the "offmassshallness" on the only fermion which interacts before and after. The Lorentz invariance-cluster separability requirement is satisfied by applying suitable Lorentz boosts on the two-body transition operators.

The main difference between our approach and Gross' approach comes from the fact that we are (presently) interested by the (123) bound states only, to be computed principally by using a single 3D equation (such as the one from which Faddeev starts). Instead of working with the two-body transition operators, we work with the two-body potentials. We can however present our approach in terms of the two-body transition operators. Our approximation of each two-body transition operator is then unique and does not depend on the operators which come in front and behind it in the expansion of the Faddeev equations. This enables us to describe our model by a single 3D potential equation, by a set of three Faddeev equations, or by the expansion of the three-body transition operator in terms of the two-body ones. Another feature of our model is to not introduce Lorentz boosts by hand: the Lorentz-invariance / cluster separability requirement is exactly satisfied if we do not truncate our potentials (practically, this means that the satisfaction of this requirement and the approximation of the potentials can be improved together). Let us go back to the three-fermion Bethe-Salpeter equation and neglect for simplicity the three-body irreducible kernel K_{123}. Defining

$$\Phi_{12} \,=\, G_1^0 G_2^0 \, K_{12} \, \Phi, \cdots \qquad \Phi \,=\, \Phi_{12} \,+\, \Phi_{23} \,+\, \Phi_{31} \,, \tag{140}$$

we get

$$(1 \,-\, G_1^0 G_2^0 \, K_{12}) \, \Phi \,=\, \Phi_{23} \,+\, \Phi_{31} \tag{141}$$

$$\Phi \,=\, (1 \,+\, G_1^0 G_2^0 \, T_{12}) \, (\, \Phi_{23} \,+\, \Phi_{31} \,) \tag{142}$$

$$\Phi_{12} \,=\, G_1^0 G_2^0 \, T_{12} \, (\, \Phi_{23} \,+\, \Phi_{31} \,). \tag{143}$$

Writing then

$$\Phi_{12} \,=\, G_1^0 G_2^0 G_3^0 \, \beta_1 \beta_2 \beta_3 \, \phi_{12}, \cdots \tag{144}$$

in order to factor out the propagators, we get

$$\phi_{12} \,=\, \beta_1 \beta_2 \, T_{12} \, G_1^0 G_2^0 \, \beta_1 \beta_2 \, (\, \phi_{23} \,+\, \phi_{31} \,). \tag{145}$$

The (12) transition matrix element corresponding to our approximation (54) of K_{12} is

$$T_{12}(\, p'_{120}, \, p_{120}, \, P_0 - p_{30} \,) \,\approx\, T_{12}^0(p_{30})$$

$$\equiv \; K_{12}^{T++}(s_{12}, s_{12}, P_0 - h_3)(1 - G_1^0 G_2^0 K_{12}^{T++}(s_{12}, s_{12}, P_0 - h_3))^{-1}. \tag{146}$$

This $T_{12}^0(p_{30})$ is analytical in the $\mathrm{Im}(p_{30}) < 0$ half plane and

$$T_{12}^0(h_3) \; = \; T_{12}^{++}(s_{12}, s_{12}, P_0 - h_3). \tag{147}$$

This approximation, combined with equation (145), implies that $\phi_{12}(p_{120}, p_{30})$ is independent of p_{120} (let us write thus $\phi_{12}(p_{30})$). Equation (145) becomes then

$$\phi_{12}(p_{30}) \; = \; \beta_1 \beta_2 \, T_{12}^0(p_{30}) \int d\, p_{120} \, G_1^0 G_2^0 \, \beta_1 \beta_2 \, [\, \phi_{23}(p_{10}) \; + \; \phi_{31}(p_{20}) \,] \tag{148}$$

where p_{10}, p_{20} must be written in terms of P_0, p_{30}, p_{120}. Equation (148), together with similar equations for $\phi_{23}(p_{10})$ and $\phi_{31}(p_{20})$, admits solutions which are analytical in the $\mathrm{Im}(p_{k0}) < 0$ half planes. At $p_{k0} = h_k$, we get the 3D Faddeev equations

$$\phi_{12}(h_3) \; = \; T_{12}^{3D++}(P_0 - h_3) \, \frac{1}{P_0 - S + i\epsilon P_0} \, [\, \phi_{23}(h_1) \; + \; \phi_{31}(h_2) \,] \, , \cdots \tag{149}$$

which can easily be transformed back into our basic three-cluster potential equation (by performing in the reverse order the transformations made above at the 4D level). Note that we would get the same result by approaching directly T_{12} by $T_{12}^0(h_3)$ instead of $T_{12}^0(p_{30})$. This means approaching K_{12} by a given function $K_{12}^0(p_{30})$, analytical in the $\mathrm{Im}(p_{30}) < 0$ half plane and equal to $K_{12}^{T++}(s_{12}, s_{12}, P_0 - h_3)$ at $p_{30} = h_3$.

The key point of the manipulations above is the dominance of the positive-energy poles of $G_1^0 G_2^0$ in (145). This was obtained by approaching K_{12}^T (or T_{12}) by a constant with Λ_{12}^{++} positive-energy projectors. We could try to insure the dominance of these two poles more economically. Let us come back to equation (145) without making any approximation on T_{12}. The elements of (145) are then the operator and functions

$$T_{12} \, (\, p'_{120}, \, p_{120}, \, P_0 - p_{30}) \tag{150}$$

$$\phi_{12} \, (\, p'_{120}, \, p_{30} \,), \quad \phi_{23} \, (\, p_{230}, \, p_{10} \,), \quad \phi_{31} \, (\, p_{310}, \, p_{20} \,) \tag{151}$$

and (145) must be integrated with respect to p_{120}, with p_{30} fixed. We must thus write p_{230}, p_{10}, p_{310}, p_{20} in terms of p_{120}, p_{30}. Searching as above for solutions $\phi_{ij} \, (\, p'_{ij0}, \, p_{k0} \,)$ which are analytical in the $\mathrm{Im}(p_{k0}) < 0$ half planes, we shall close our integration path clockwise (counterclockwise) in front of ϕ_{23} (ϕ_{31}) and keep only the residue of the pole of $\Lambda_{12}^{++} G_1^0 G_2^0$ which puts fermion 1 (2) on its positive-energy mass shell. The elements of (145) are then replaced by

$$T_{12} \, (\, p'_{120}, \, p_{120}, \, P_0 - p_{30}) \; \rightarrow \; T_{12} \, (\, p'_{120}, \, s_{12} \, \mp \, \frac{P_0 - S}{2} \, \pm \, \frac{p_{30} - h_3}{2}, \, P_0 - p_{30}) \tag{152}$$

$$\int d\, p_{120} \, G_1^0 G_2^0 \, \beta_1 \beta_2 \; \rightarrow \; \frac{-2i\pi \, \Lambda_{12}^{++}}{(P_0 - S) - (p_{30} - h_3) + i\epsilon} \tag{153}$$

$$\phi_{23} \, (\, p_{230}, \, p_{10} \,) \; \rightarrow \; \phi_{23} \, (\, s_{23} \, + \, \frac{P_0 - S}{2} \, - \, (\, p_{30} - h_3 \,), \, E_1 \,) \tag{154}$$

$$\phi_{31} \, (\, p_{310}, \, p_{20} \,) \; \rightarrow \; \phi_{31} \, (\, s_{31} \, - \, \frac{P_0 - S}{2} \, + \, (\, p_{30} - h_3 \,), \, E_2 \,) \tag{155}$$

where we take the upper signs in (152) in front of ϕ_{23}, the lower signs in front of ϕ_{31}.

The manipulations above are submitted to some restrictions on the T_{ij}. In the p_{k0} variable the T_{ij} must be analytical in the $\mathrm{Im}(p_{k0}) < 0$ half planes. In the p'_{ij0} and p_{ij0} variables the T_{ij} must be analytical in the whole complex plane. Moreover, the T_{ij} must also be asymptotically bounded in the three variables and contain Λ_{ij}^{++} positive-energy projectors. These conditions are of course not satisfied by the exact transition matrix elements, but we shall assume that the singularities and the negative-energy parts of the T_{ij} can be neglected in the computation of the integrals with respect to the relative energies.

Taking the equation for ϕ_{12} at $p_{30} = E_3$ and $p'_{120} = s_{12}^{\pm} = s_{12} \pm \frac{1}{2}(P_0 - S)$, means that we put fermions 2 and 3 (for +) or fermions 1 and 3 (for -) on their mass shells. This leads to

$$\phi_{12}^2 \; = \; -2i\pi \, \beta_1 \beta_2 \, (\, T_{12}^{2,1} \, \frac{1}{P_0 - S + i\epsilon} \, \phi_{23}^2 \, + \, T_{12}^{2,2} \, \frac{1}{P_0 - S + i\epsilon} \, \phi_{31}^1 \,) \tag{156}$$

$$\phi_{12}^1 \;=\; -2i\pi\,\beta_1\beta_2\,(\,T_{12}^{1,1}\,\frac{1}{P_0-S+i\epsilon}\;\phi_{23}^2\;+\;T_{12}^{1,2}\,\frac{1}{P_0-S+i\epsilon}\;\phi_{31}^1\,) \tag{157}$$

where the upper indexes refer to the initial and final on mass shell non-spectator fermions. Equations (156),(157), together with similar equations for the (23) and (31) pairs, are a closed system of six 3D equations. In each T_{ij} two fermions are on the mass shell: the spectator fermion k and the fermion which is not going to interact next left (p'_{ij0}) or next right (p_{ij0}). This is of course the philosophy of Gross' spectator model [17, 18].

If we replace $s^{\pm}$ by s in the T_{ij} we get our basic three-cluster model. This supplementary approximation can be justified by noticing that we already neglected the contributions of the singularities of T_{ij} in the p'_{ij0} and p_{ij0} complex planes. The dependence on these variables can thus not be very strong, as we know that a function which is analytical and bounded on the whole complex sphere must necessarily be a constant. Using the same argument of the consistency of the approximations, we could also argue that T_{ij} constant implies K_{ij}^T and K_{ij} constant with $K_{ij}^T = K_{ij}$. Our basic three-cluster equation is thus not a priori a worse approximation of the three-fermion Bethe-Salpeter equation than Gross' equations, nor a better approximation than the simpler positive-energy instantaneous approximation or the corresponding Born approximation. It reflects the choice of preserving the exact cluster-separated limits.

To conclude the comparison between our 3D reduction and Gross' spectator model, we can say that, at lowest order, our 3D reduction is much simpler: our two-body transition operators are always taken at the same value of the corresponding relative energie, so that we can transform the set of coupled equations for the transition operators into a unique equation for a wave function. We do not use Lorentz boosts, as we choose to exploit the implicit Lorentz covariance of the unapproximated separated-clusters limit equations, so that the effects of these boosts will be approached in principle by the progressive introduction of the higher-order two-fermion terms. Gross' higher-order corrections introduced by taking the transition operators with different fermions on the mass shell are not a priori more important than the contributions of the singularities of these transition operators themselves to the closed path integrals in the relative energies. Our brief calculation with the one-photon exchange graph (subsection 5.3) shows that, in this case, the contributions of these singularities are in fact more important, by an order of α (but it turns out that the leading order of these contributions will be cancelled by the leading order contribution of the first crossed graph [42]). Our higher-order calculation takes both types of contributions into account.

7 Heavy mass limits and external potentials.

7.1 Heavy mass limits in the two-fermion problem.

When the mass of one of the fermions goes to infinity, we must find the equation of the other fermion in a potential (a Dirac equation with a Coulomb potential in QED). This limit can be obtained directly, or via a rearrangement of the equation. With Breit's equation with the second-order constraint and with Sazdjian's equation, the higher-order ladder and crossed terms cancel mutually at the one-body limit, so that the correct limit of the potential is already contained in the Born term. With Salpeter's equation with the second-order constraint, and with the positive-energy equation with a second-order constraint, the limit is also a Dirac-Coulomb equation, but solved with respect to the $\Lambda^+\psi$ part of the wave function: the $\Lambda^-\psi$ part is transformed into higher-order contributions to the potential. The physical content remains identical to that of a Dirac-Coulomb equation, but the correct limit of the potential is no more entierely contained in the Born term, so that a truncation of the potential would spoil the one-body limit [8-12]. With the first-order constraint, the results are similar, but after complicated rearrangements of the equation.

In the three-fermion problem, we expect similarly that, at the limit of an infinite fermion mass, we would get an equation describing two fermions in an external potential. It is thus interesting to compare this equation with the equation we wrote directly [16] for the two-fermion in an external potential system.

7.2 Two fermions in an external potential.

The two-fermion plus potential problem can be approached in two ways which we shall call the two-body and the three-body approaches. In the two-body approach, the external potential is included in the definition of the creation and annihilation operators of field theory. Practically, we can keep the equations

obtained in the treatment of the pure two-fermion problem, using simply a generalized definition of the h_i [14, 15, 16]:

$$h_i = \vec{\alpha}_i \,.\vec{p}_i + \beta_i \, m_i \, + \, V_i(\vec{x}_i) \tag{158}$$

where V_i is the external potential acting on the fermion i. All quantities can be expanded on the basis built with the eigenstates of h_1 and h_2. In the three-body approach, the free creation and annihilation operators are used and the external potential is treated as an heavy third body. This approach will be presented as an heavy mass limit of the three fermion problem in the next subsection.

The equations for two fermions in an external potential must be written in the laboratory reference frame (in which the external potential is defined) and are not Lorentz invariant. If we "switch off" the mutual interaction we get a pair of uncoupled equations for two independent fermions in an external potential. If we switch off the external potential the equations remain written in the laboratory frame, which still refers to the vanished external potential. If the reduction series is not truncated, this no-external interaction limit of the equation is equivalent to a pure two-fermion covariant Bethe-Salpeter equation.

7.3 Heavy mass limits in the three-fermion problem.

Let us now take our 3D equation (42), with the potentials V_{ij} obtained by performing 3D reductions based on the $G_{ij}^{\delta P}$ as in section 2, but with the second-order constraints $\delta(p_{ij0} - \mu_{ij})$ instead of the first-order constraints $\delta(p_{ij0} - s_{ij})$. When m_3, for example, goes to infinity (two-body limit), we get, writing $P_0 = W_{12} + m_3$:

$$\psi \, = \, \frac{1}{W_{12} - h_1 - h_2} \, (V_{12} + V_2^+ + V_1^+) \, \psi \tag{159}$$

with $(P_0 - h_3)$ replaced by W_{12} in V_{12}. The potential V_2^+ is given by the series

$$V_2^+ \, = \, \Lambda_2^+ \left[V_2 \, + \, V_2 \, \frac{\Lambda_2^-}{(W_{12} - h_1) + E_2 - i\epsilon} \, V_2 \, + \, ... \right] \Lambda_2^+ \tag{160}$$

where V_2 is an external potential acting on fermion 2 (it is a Coulomb potential in QED, if we use the second-order constraint, but in general it could still depend on $(W_{12} - h_1)$). The potential V_1^+ is given by a similar formula.
The differences with the two-body approach of the two-body plus potential problem are:

– The projectors $\Lambda_i^{\pm}$ are the free ones.
– The external potential for fermion 2 is now V_2^+, given by the series (160). This is the potential obtained by solving the equation for fermion 2 in the external potential V_2 with respect to the positive free energy components, the unobservable energy of fermion 2 being replaced in it by $W_{12} - h_1$.
– V_2 itself could still depend on $(W_{12} - h_1)$ and vice-versa.

The cluster separability property in the three-body way survives the high-mass limit: switching off V_2 and V_1 leads to the equation for the (12) two-fermion system, switching off V_{12} and V_1 leads to a free Dirac equation for fermion 1 with the Λ_2^+ projection of the Dirac equation for fermion 2 in the external potential V_2. In the two-body approach of the two-fermion plus potential problem we had however more: switching off the mutual interaction only led to a pair of independent equations for each fermion in the external potential. Here, in the three-body approach, the equation does not split perfectly into two parts: we can write $W_{12} = W_1 + W_2$, but we have $W_{12} - h_1$ instead of W_2 in the series defining V_2^+ and vice-versa. This is a consequence of the energy dependence introduced into (160) by the use of the anti-continuum dissolution projectors, combined with our choice of putting the spectator fermion on the mass shell in each two-body interaction, and neglecting the three-body terms which could balance this modification. The discrepancy is of order V^4. At the two-fermion plus potential level, a better but still not perfect equation would be obtained by replacing h_1 by $h_1 + V_1$ in the series defining V_2^+ and vice-versa.

8 Conclusions

We have written a 3D equation for three fermions (our basic three-cluster equation) by combining the three two-body potentials obtained by an untruncated propagator-based 3D reduction of the corresponding

two-fermion Bethe-Salpeter equations, putting the spectator fermion on the mass shell. In this way, the cluster-separated limits are exact, and the Lorentz invariance / cluster separability requirement is automatically satisfied, provided no supplementary approximation, like the Born approximation, is made. Such a truncation would render the Lorentz covariance of the two-fermion clusters only approximate (a Born approximation preserving the Lorentz invariance / cluster separability property can be obtained by using another 3D reduction based on a covariant second-order two-body propagator of Sazdjian, combined with a covariant substitute of Λ^{++}. This leads to a 3D three-cluster equation which is covariantly Born approximable, but more complicated [40]). Our 3D equation can be written in terms of the two-body potentials, or in terms of two-body transition operators (Faddeev formalism). The use of positive free-energy projectors in the chosen reductions of the two-fermion Bethe-Salpeter equations prevents our 3D three-fermion equation from continuum dissolution. The potentials are hermitian and depend only slowly on the total three-fermion energy. The one high-mass limits of our "basic three-cluster equation" are approximately exact. The correction of the remaining discrepancy would demand the introduction of higher-order three-body terms.

Our combination of cluster separability and Lorentz invariance in the three fermion problem makes explicit use of the fact that the clusters can only be two-fermion and/or free fermion states. This is not directly adaptable to four or more fermion systems. In this respect, 3 is still not N.

Our 3D equation can be written directly, or derived from an approximation of the three-fermion Bethe-Salpeter equation, in which the three-body kernel is neglected while the two-body kernels are approached by positive-energy instantaneous expressions, with the spectator fermion on the mass shell. The correction terms are thus known at the Bethe-Salpeter level and can be transformed into corrections to the 3D equation.

Our lowest-order three-fermion 3D equation is much simpler than the equations written in the framework of Gross' spectator model, which contain some induced higher-order three-fermion contributions. Our calculation of the next order shows that these contributions are not a priori more important that the ones coming from the singularities of the kernels, neglected in Gross' model. We think that the relative importance of the different contributions could largely depend on the specific problem studied.

There exists an infinity of ways of performing the 3D reduction of the two-fermion Bethe-Salpeter equation. The potentials generated by these reductions could all by used to build a three-fermion 3D equation, keeping however in mind the continuum dissolution problem. This leaves us a large freedom to suit phenomenological needs.

We have also examined the possibility of using kernel-based 3D reductions in the two and three-fermion problems. We found that this kind of 3D reduction can also be used in the calculation of bound states, despite the fact that they do not lead to the correct scattering operators (excepting in Phillips and Wallace's reduction).

As a first test of our methods in a specific problem, we are presently working on a system of two or three fermions exchanging photons in Feynman or Coulomb's gauges [42]. The various terms generated by our propagator-based 3D reduction will be compared with these generated by a kernel-based 3D reduction and with these written in the framework of Gross' spectator model.

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